Unimodal例外型特異点における代数的局所コホモロジー類 (微分方程式の漸近解析と超局所解析)

中村, 弥生; 田島, 慎一

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Unimodal 例外型特異点 における 代数的局所コホモロジー類

中村 弥生 (Yayoi Nakamura) お茶の水女子大学大学院*
田島 慎一 (Shinichi Tajima) 新潟大学工学部情報工学科†

1 序

孤立特異点を持つ擬斉次多項式 $f=f(x_1,\ldots,x_n)$ に対し, 原点に台を持つ代数的局所コホモロジー類 $[1/f_{x_1}\cdots f_{x_n}]\in\mathcal{H}^n_{[0]}(\mathcal{O}_X)$ を考える. 但し, $f_{x_j}=\partial f/\partial x_j,\ j=1,\ldots,n$ である. コホモロジー類 $[1/f_{x_1}\cdots f_{x_n}]$ の annihilator について, 次が成り立つ.

Fact :$f=f(x_1,\ldots,x_n)$ を quasiweight $(\alpha_1,\ldots,\alpha_n)$, quasidegree d_w の n 変数擬斉次多項式とする. $f_{x_1}=\partial f/\partial x_1,\ \ldots,\ f_{x_n}=\partial f/\partial x_n$ に対し, 原点に台を持つ代数的局所コホモロジー類 $[1/f_{x_1}\cdots f_{x_n}]$ を考える. $[1/f_{x_1}\cdots f_{x_n}]$ の微分加群としての annihilating ideal を $\mathcal{A}nn$ とおくと,

$$\mathcal{A}nn=\langle f_{x_1},\ldots,f_{x_n},P\rangle$$

が成り立つ. 但し, $P=\alpha_1x_1\frac{\partial}{\partial x_1}+\cdots+\alpha_nx_n\frac{\partial}{\partial x_n}+nd_w-(\alpha_1+\cdots+\alpha_n)$ である.

($\mathcal{A}nn$ のグレブナ基底については [2] を参照されたい.)

この事実を用いると, 微分作用素 $f_{x_1},\ldots,f_{x_n}$, P を用いて代数的局所コホモロジー類 $[1/f_{x_1}\cdots f_{x_n}]$ を特徴付けることができる ([4], [5] 参照). つまり, f が擬斉次多項式の場合, 代数的局所コホモロジー類 $[1/f_{x_1}\cdots f_{x_n}]$ を特徴付ける微分作用素は euler 型で与えることができる. それに比べ, f が半擬斉次多項式の場合, 一般に, 1 階の微分作用素では代数的局所コホモロジー類を特徴付けることはできない.

我々は, V.I.Arnol'd による分類 ([1]) に従い, Unimodal 例外型孤立特異点 (E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}, Q_{10}, Q_{11}, Q_{12}, S_{11}, S_{12}, U_{12}) に付随する代数的局所コホモロジー類の annihilator を計算した. その結果, これらの場合, 代数的局所コホモロジー類を特徴付けるには 2 階の微分作用素が必要となることが明らかになった.

本稿では, まず初めに, §1 で E_{12} 型特異点に対してコホモロジー類の計算法を述べ, §2 で主結果を与える. §3 で Unimodal 特異点に関する計算結果を与え, 最後に §4 で Bimodal 特異点である E_{18} 型の場合について述べる.

2 代数的局所コホモロジー類の計算例 (E_{12} 型)

$X=\mathbb{C}^2$ 上の半擬斉次多項式 $f(x,y)=x^3+xy^5+y^7$ で与えられる E_{12} 型 Unimodal singularity を考える. $f_x=\partial f/\partial x=3x^2+y^5$, $f_y=\partial f/\partial y=5xy^4+7y^6$ とおく. f_x, f_y で生成される $\mathcal{O}_X$ 上のイデアル $I=\langle f_x,f_y\rangle$ に対し, 全次数辞書式順序 $x\succ y$ でのグレブナ基底は $\mathrm{Gb}=\{5x^3+7y^2x^2,125x^4-1029yx^3,3x^2+y^5,-21yx^2+5y^4x\}$ で与えられる. I の準素イデアル分解は $I=I_1\cap I_2$, $I_1=\langle 25y+147,3125x+151263\rangle$, $I_2=\langle y^8,5y^4x+7y^6,3x^2+y^5\rangle$ で与えられる. $Y=\{(x,y)\in X\mid f_x=f_y=0\}$ とおくと, $Y=\{3125x+151263=25y+147=0\}\cup\{x=y=0\}$ であり, 原点の重複度は 12 である.

Y に台を持つ代数的局所コホモロジー類 $[1/f_xf_y]$ に対し, $[1/f_xf_y]$ を annihilate する高々j 階の微分作用素の生成する左イデアルを $\mathcal{A}nn^{(j)}$ とおく. $\mathcal{A}nn^{(1)}$ を求めると, $\mathcal{A}nn^{(1)}=\langle f_x,f_y,P_1,P_2\rangle$ を得る. 但し,
$P_1=(5xy+7y^3)\partial_y+20x+42y^2$,
$P_2=(3500y^2+2058y)x\partial_x+((-1000y-735)x+7203y^2)\partial_y-4000x+7700y^2+84378y$
($\partial_x=\partial/\partial x$, $\partial_y=\partial/\partial y$) である. これらの作用素を用いて原点に台を持つ代数的局所コホモロジー類 $\sigma=[1/f_xf_y]|_{(0,0)}$ の表現を計算する ([4] 参照). 方程式 $f_x\sigma=f_y\sigma=P_1\sigma=P_2\sigma=0$, $|\partial(f_x,f_y)/\partial(x,y)|\sigma=12[1/xy]$ を解くことにより, σ の表現

*nakamura@math.ocha.ac.jp
†tajima@geb.ge.niigata-u.ac.jp

$$
\begin{aligned}
\sigma = [\ & \boxed{?}\frac{1}{xy} - \frac{1220703125}{1483273860320763}\frac{1}{xy^2} + \frac{48828125}{10090298369529}\frac{1}{xy^3} - \frac{1953125}{68641485507}\frac{1}{xy^4} \\
& + \frac{78125}{466948881}\frac{1}{xy^5} - \frac{3125}{3176523}\frac{1}{xy^6} + \frac{125}{21609}\frac{1}{xy^7} - \frac{5}{147}\frac{1}{xy^8} - \frac{9765625}{1441471195647}\frac{1}{x^2y} \\
& + \frac{390625}{9805926501}\frac{1}{x^2y^2} - \frac{15625}{66706983}\frac{1}{x^2y^3} + \frac{625}{453789}\frac{1}{x^2y^4} - \frac{25}{3087}\frac{1}{x^2y^5} + \frac{1}{21}\frac{1}{x^2y^6} \\
& + \frac{3125}{9529569}\frac{1}{x^3y} - \frac{125}{64827}\frac{1}{xy^2} + \frac{5}{441}\frac{1}{x^3y^3} - \frac{1}{63}\frac{1}{x^4y}\ ]
\end{aligned}
$$

を得る. つまり, $\mathcal{A}nn^{(1)}$ を用いた計算では, デルタ関数に相当する部分 $[1/xy]$ の係数が決まらないことが分かる. また, $\mathcal{A}nn^{(1)}$ を原点に局所化したものは, $\langle y^8, 5y^4x + 7y^6, 3x^2 + y^5, P_{0,1}, P_{0,2}\rangle$ となる. 但し,

$$
\begin{aligned}
P_{0,1} = \ & (5yx + 7y^3)\partial_y + 20x + 42y^2 \\
P_{0,2} = \ & 470880590578020yx\partial_x + (-16817163949215x + 164808206702307y^2)\partial_y \\
& -9765625000x^3 + (-80390625000y + 236348437500)x^2 \\
& +(56273437500y^3 - 330887812500y^2 + 1945620337500y - 11440247584500)x \\
& +463242937500y^4 - 2723868472500y^3 + 16016346618300y^2 + 1930610421369882y
\end{aligned}
$$

である. $P_{0,1}$, $P_{0,2}$ の形から, ホロノミック系 $\mathcal{D}_X/\mathcal{A}nn^{(1)}$ の原点における重複度が 2 であることが分かる.

次に, $\mathcal{A}nn^{(2)}$ を求めると, $\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$ を得る. 但し,

$$
\begin{aligned}
P = \ & ((68250000y - 24863426875)x^2 + (-37182906425y^2 - 10656175824y)x)\partial_x^2 \\
& +(-8437500x^2 + (637980000y + 694297170)x - 3708061182y^2)\partial_y\partial_x \\
& +((-885281250y - 128629930625)x - 179156250y^3 - 111958930125y^2 - 54216894564y)\partial_x \\
& +(-18742500x + 154288260y)\partial_y^2 \\
& +(-25312500x - 511875000y^2 - 2118711000y + 2082891510)\partial_y - 5545968750y - 102242340375
\end{aligned}
$$

である. これらの作用素を用いて σ の表現を計算すると,

$$
\begin{aligned}
\sigma = [\ & \frac{30517578125}{218041257467152161}\frac{1}{xy} - \frac{1220703125}{1483273860320763}\frac{1}{xy^2} + \frac{48828125}{10090298369529}\frac{1}{xy^3} - \frac{1953125}{68641485507}\frac{1}{xy^4} \\
& + \frac{78125}{466948881}\frac{1}{xy^5} - \frac{3125}{3176523}\frac{1}{xy^6} + \frac{125}{21609}\frac{1}{xy^7} - \frac{5}{147}\frac{1}{xy^8} - \frac{9765625}{1441471195647}\frac{1}{x^2y} \\
& + \frac{390625}{9805926501}\frac{1}{x^2y^2} - \frac{15625}{66706983}\frac{1}{x^2y^3} + \frac{625}{453789}\frac{1}{x^2y^4} - \frac{25}{3087}\frac{1}{x^2y^5} + \frac{1}{21}\frac{1}{x^2y^6} \\
& + \frac{3125}{9529569}\frac{1}{x^3y} - \frac{125}{64827}\frac{1}{xy^2} + \frac{5}{441}\frac{1}{x^3y^3} - \frac{1}{63}\frac{1}{x^4y}\ ]
\end{aligned}
$$

を得る.

また, $\mathcal{A}nn^{(2)}$ を原点に局所化したものは, $\langle y^8, 5y^4x + 7y^6, 3x^2 + y^5, P_0\rangle$ となる. 但し,

$$
\begin{aligned}
P_0 = \ & (620690779589826484980x + 427360864635618235556y^2)\partial_x\partial_y \\
& +(328792372371102465 00x + 16068800153474932500y^3 - 94484544902432603100y^2 \\
& \quad +256416518781370941336y)\partial_x \\
& +305257760454013025400y\partial_y^2 \\
& +((-131562847221750000 0y + 1439969663151534375)x \\
& \quad -8814296054882311875y^2 + 10382917022245341000 0y + 336801062367594371358 0)\partial_y \\
& -1165771484375000x^3 + (-2060009765625000y + 116844758789062500)x^2 \\
& +(-3548743652343750 0y^3 + 14455661308593750 0y^2 \\
& \quad -4730289445546875 00y - 4697651664385312500)x \\
& +524043573046875 00y^4 + 73033723418906250 0y^3 - 48749755019450625 00y^2 \\
& -2080703609616625875 0y + 500110503238150591500
\end{aligned}
$$

である. イデアル $\langle y^8, 5y^4x + 7y^6, 3x^2 + y^5, P_0\rangle$ のグレブナ基底を計算することにより, ホロノミック系 $\mathcal{D}_X/\mathcal{A}nn^{(2)}$ の原点における重複度が 1 であることが分かる. 従って, E_{12} の場合, $\mathcal{A}nn = \mathcal{A}nn^{(2)}$ が成り立つ.

なお, 微分作用素 P_1, P_2, P 等の構成法については, 文献 [3] を参照されたい.

3 主結果

Unimodal 例外型特異点に対し, 代数的局所コホモロジー類を特徴付ける微分作用素について調べる. 本稿では, quasihomogeneous とならない以下の標準形について計算する.

$$
\begin{array}{llcl}
2\text{ 変数} & E_{12} & : & f(x,y) = x^3 + y^7 + axy^5 \\
 & E_{13} & : & f(x,y) = x^3 + xy^5 + ay^8 \\
 & E_{14} & : & f(x,y) = x^3 + y^8 + axy^6 \\
 & Z_{11} & : & f(x,y) = x^3y + y^5 + axy^4 \\
 & Z_{12} & : & f(x,y) = x^3y + xy^4 + ax^2y^3 \\
 & Z_{13} & : & f(x,y) = x^3y + y^6 + axy^5 \\
 & W_{12} & : & f(x,y) = x^4 + y^5 + ax^2y^3 \\
 & W_{13} & : & f(x,y) = x^4 + xy^4 + ay^6 \\
3\text{ 変数} & Q_{10} & : & f(x,y,z) = x^3 + y^4 + yz^2 + axy^3 \\
 & Q_{11} & : & f(x,y,z) = x^3 + y^2z + xz^3 + az^5 \\
 & Q_{12} & : & f(x,y,z) = x^3 + y^5 + yz^2 + axy^4 \\
 & S_{11} & : & f(x,y,z) = x^4 + y^2z + xz^2 + ax^3z \\
 & S_{12} & : & f(x,y,z) = x^2y + y^2z + xz^3 + az^5 \\
 & U_{12} & : & f(x,y,z) = x^3 + y^3 + z^4 + axyz^2
\end{array}
$$

$(x,y,z) \in X = \mathbb{C}^3$ に対し, $f_x = \partial f(x,y,z)/\partial x$, $f_y = \partial f(x,y,z)/\partial y$, $f_z = \partial f(x,y,z)/\partial z$ とおく. f_x, f_y, f_z で生成されるイデアル $I = \langle f_x, f_y, f_z \rangle$ に対し, $Y = V(I)$ に台を持つ代数的局所コホモロジー類 $[1/f_xf_yf_z]$ の $\mathcal{D}_X$ 上の annihilating ideal を $\mathcal{A}nn$ とおく. 但し, $\mathcal{D}_X$ は X 上の線形偏微分作用素の層とする. また, $[1/f_xf_yf_z]$ を annihilate する高々 j 階の線形偏微分作用素の生成する左イデアルを $\mathcal{A}nn^{(j)}$ とおく. σ を, 原点に台を持つ代数的局所コホモロジー類 $[1/f_xf_yf_z]|_{(0,0,0)}$ とする. $X = \mathbb{C}^2 \ni (x,y)$ の場合も同様の記号を用いる (§2 参照).

次の結果を得た.

Proposition 1

(i) ホロノミック系 $\mathcal{D}_X/\mathcal{A}nn^{(1)}$ の原点における重複度 $= 2$

(ii) ホロノミック系 $\mathcal{D}_X/\mathcal{A}nn^{(2)}$ の原点における重複度 $= 1$

Theorem 2 $\mathcal{A}nn^{(2)} = \mathcal{A}nn$

Theorem 3

(i) $Hom_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{A}nn^{(1)}, \mathcal{H}^{\dim X}_{[0]}(\mathcal{O}_X)) = \mathrm{Span}\{\delta_0, \sigma\}$,
ここで, δ_0 は原点に台を持つデルタ関数 ($\dim X = 3$ の場合は $[1/xyz]$, $\dim X = 2$ の場合は $[1/xy]$) である.

(ii) $Hom_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{A}nn^{(2)}, \mathcal{H}^{\dim X}_{[0]}(\mathcal{O}_X)) = \mathrm{Span}\{\sigma\}$.

4 具体的計算結果

前節にあげた定理に関する具体的計算結果（コホモロジー類の表現や $\mathcal{A}nn^{(1)}$, $\mathcal{A}nn^{(2)}$ の生成元）を, 下に挙げる標準形で与えられる Unimodal 特異点に対し, それぞれまとめておく. 標準形にはパラメーター a が含まれるが, 本稿では $a = 1$ として計算を行った. グレブナ基底等の計算は, $x \succ y \succ z$ (2 変数の場合は $x \succ y$) として, 全次数辞書式順序を用いて行った. 計算に用いた数式処理システム (kan/sm1, risa/asir) については [6] を参照されたい.

$$
\begin{array}{llcl}
2\,\text{変数} & E_{12} & : & f(x,y) = x^3 + y^7 + xy^5 \\
 & E_{13} & : & f(x,y) = x^3 + xy^5 + y^8 \\
 & E_{14} & : & f(x,y) = x^3 + y^8 + xy^6 \\
 & Z_{11} & : & f(x,y) = x^3y + y^5 + xy^4 \\
 & Z_{12} & : & f(x,y) = x^3y + xy^4 + x^2y^3 \\
 & Z_{13} & : & f(x,y) = x^3y + y^6 + xy^5 \\
 & W_{12} & : & f(x,y) = x^4 + y^5 + x^2y^3 \\
 & W_{13} & : & f(x,y) = x^4 + xy^4 + y^6 \\
3\,\text{変数} & Q_{10} & : & f(x,y,z) = x^3 + y^4 + yz^2 + xy^3 \\
 & Q_{11} & : & f(x,y,z) = x^3 + y^2z + xz^3 + z^5 \\
 & Q_{12} & : & f(x,y,z) = x^3 + y^5 + yz^2 + xy^4 \\
 & S_{11} & : & f(x,y,z) = x^4 + y^2z + xz^2 + x^3z \\
 & S_{12} & : & f(x,y,z) = x^2y + y^2z + xz^3 + z^5 \\
 & U_{12} & : & f(x,y,z) = x^3 + y^3 + z^4 + xyz^2
\end{array}
$$

4.1 E_{12} 型特異点

$f = x^3 + y^7 + xy^5$,
$f_x = 3x^2 + y^5$,$f_y = 7y^6 + 5xy^4$.
$\mathrm{Gb} = \{5x^3 + 7y^2x^2, 125x^4 - 1029yx^3, 3x^2 + y^5, -21yx^2 + 5y^4x\}$.
$I = I_1 \cap I_2$,
$I_1 = \langle 25y + 147, 3125x + 151263\rangle$, $\sqrt{I_1} = \langle 25y + 147, 3125x + 151263\rangle$,
$I_2 = \langle y^8, 5y^4x + 7y^6, 3x^2 + y^5\rangle$, $\sqrt{I_2} = \langle y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, P_1, P_2\rangle$,
$P_1 = (5yx + 7y^3)\partial_y + 20x + 42y^2$,
$P_2 = (3500y^2 + 2058y)x\partial_x + ((-1000y - 735)x + 7203y^2)\partial_y - 4000x + 7700y^2 + 84378y$.

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$,

$$
\begin{aligned}
P = \ & ((68250000y - 24863426875)x^2 + (-37182906425y^2 - 10656175824y)x)\partial_x^2 \\
& +(-8437500x^2 + (637980000y + 694297170)x - 3708061182y^2)\partial_y\partial_x \\
& +((-885281250y - 128629930625)x - 179156250y^3 - 111958930125y^2 - 54216894564y)\partial_x \\
& +(-18742500x + 154288260y)\partial_y^2 \\
& +(-25312500x - 511875000y^2 - 2118711000y + 2082891510)\partial_y \\
& -5545968750y - 102242340375.
\end{aligned}
$$

$$
\begin{aligned}
\sigma = [\ & \frac{5^{15}}{3^8 7^{16}}\frac{1}{xy} - \frac{5^{13}}{3^7 7^{14}}\frac{1}{xy^2} + \frac{5^{11}}{3^6 7^{12}}\frac{1}{xy^3} - \frac{5^9}{3^5 7^{10}}\frac{1}{xy^4} + \frac{5^7}{3^4 7^8}\frac{1}{xy^5} - \frac{5^5}{3^3 7^6}\frac{1}{xy^6} + \frac{5^3}{3^2 7^4}\frac{1}{xy^7} \\
& -\frac{5}{3\cdot 7^2}\frac{1}{xy^8} - \frac{5^{10}}{3^6 7^{11}}\frac{1}{x^2y} + \frac{5^8}{3^5 7^9}\frac{1}{x^2y^2} - \frac{5^6}{3^4 7^7}\frac{1}{x^2y^3} + \frac{5^4}{3^3 7^5}\frac{1}{x^2y^4} - \frac{5^2}{3^2 7^3}\frac{1}{x^2y^5} + \frac{1}{3\cdot 7}\frac{1}{x^2y^6} \\
& +\frac{5^5}{3^5 7^6}\frac{1}{x^3y} - \frac{5^3}{3^4 7^4}\frac{1}{x^3y^2} + \frac{5}{3^3 7^2}\frac{1}{x^3y^3} - \frac{1}{3^3 7}\frac{1}{x^4y}\ ].
\end{aligned}
$$

4.2 E_{13} 型特異点

$f = x^3 + xy^5 + y^8$,
$f_x = 3x^2 + y^5$, $f_y = 5xy^4 + 8y^7$.
$\mathrm{Gb} = \{4608x^4 + 125yx^3, 3x^2 + y^5, -24y^2x^2 + 5y^4x, 5x^3 + 8y^3x^2, (-192y^2 - 25y)x^3\}$.
$I = I_1 \cap I_2$,
$I_1 = \langle 192y + 25, 884736x - 3125\rangle$, $\sqrt{I_1} = \langle 192y + 25, 884736x - 3125\rangle$,
$I_2 = \langle yx^3, x^4, 3x^2 + y^5, 24y^2x^2 - 5y^4x, 5x^3 + 8y^3x^2\rangle$, $\sqrt{I_2} = \langle y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, P_1, P_2\rangle$,
$P_1 = (5x^2 + 8y^3x)\partial_x + 15x + 16y^3$,
$P_2 = ((2688y^2 + 125y)x - 360y^4)\partial_x + (-240x + 50y^2)\partial_y + 5376y^2 + 575y$.

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$,

$$
\begin{aligned}
P = \ & ((7077888y-4492800)x^2+(-13639680y^3-139200y^2+71875y)x+9000y^4)\partial_x^2 \\
& +((1244160y+24000)x+28750y^2)\partial_y\partial_x \\
& +((155713536y-3041280)x-43868160y^3-288000y^2+402500y)\partial_x \\
& +(42467328y^2+6773760y+24000)\partial_y \\
& +566231040y+55710720.
\end{aligned}
$$

$$
\begin{aligned}
\sigma = [\ & -\frac{2^{48}3^8}{5^{17}}\frac{1}{xy}+\frac{2^{42}3^7}{5^{15}}\frac{1}{xy^2}-\frac{2^{36}3^6}{5^{13}}\frac{1}{xy^3}+\frac{2^{30}3^5}{5^{11}}\frac{1}{xy^4}-\frac{2^{24}3^4}{5^9}\frac{1}{xy^5}+\frac{2^{18}3^3}{5^7}\frac{1}{xy^6}-\frac{2^{12}3^2}{5^5}\frac{1}{xy^7} \\
& +\frac{2^6 3}{5^3}\frac{1}{xy^8}-\frac{1}{5}\frac{1}{xy^9}-\frac{2^{33}3^5}{5^{12}}\frac{1}{x^2y}+\frac{2^{27}3^4}{5^{10}}\frac{1}{x^2y^2}-\frac{2^{21}3^3}{5^8}\frac{1}{x^2y^3}+\frac{2^{15}3^2}{5^6}\frac{1}{x^2y^4}-\frac{2^9 3}{5^4}\frac{1}{x^2y^5} \\
& +\frac{2^3}{5^2}\frac{1}{x^2y^6}-\frac{2359296}{5^7}\frac{1}{x^3y}+\frac{12288}{5^5}\frac{1}{x^3y^2}-\frac{2^6}{5^3}\frac{1}{x^3y^3}+\frac{1}{3\cdot 5}\frac{1}{x^3y^4}-\frac{2^3}{3\cdot 5^2}\frac{1}{x^4y} \ ].
\end{aligned}
$$

4.3 E_{14} 型特異点

$f = x^3 + xy^6 + y^8$,

$f_x = 3x^2 + y^6$, $f_y = 6xy^5 + 8y^7$.

$\mathrm{Gb} = \{3x^3 + 4y^2x^2, 9yx^4 - 64yx^3, 9x^5 - 64x^4, 3x^2 + y^6, -4yx^2 + y^5x\}$.

$I = I_1 \cap I_2$,

$I_1 = \langle 9x - 64, 3y^2 + 16\rangle$, $\sqrt{I_1} = \langle 3y^2 + 16, 9x - 64\rangle$,

$I_2 = \langle 3x^3 + 4y^2x^2, yx^3, x^4, 3x^2 + y^6, 4yx^2 - y^5x\rangle$, $\sqrt{I_2} = \langle y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, P_1, P_2\rangle$,

$P_1 = (3x^2 + 4y^2x)\partial_x + 9x + 8y^2$,

$P_2 = ((9y^3 + 32y)x + 4y^5)\partial_x + (3y^4 + 16y^2)\partial_y + 42y^3 + 176y$.

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$,

$$
\begin{aligned}
P = \ & ((8064y^2+1600512)x^2+2076672y^2x)\partial_x^2 \\
& +(-3024yx^2+18432yx-4096y^3)\partial_y\partial_x \\
& +(93312x^3+(84807y^2+554688)x^2+(-7452y^4+756288y^2+7117824)x \\
& \quad +20160y^6-29184y^4+6201344y^2)\partial_x \\
& +(-6912x-9216y^2)\partial_y^2 \\
& +(31104yx^2+(28269y^3+268272y)x-2484y^5+380736y^3-368640y)\partial_y \\
& +435456x^2+(354294y^2+3050784)x-17172y^4+4063680y^2+746496.
\end{aligned}
$$

$$
\begin{aligned}
\sigma = [\ & -\frac{3^4}{2^{21}}\frac{1}{xy}+\frac{3^3}{2^{17}}\frac{1}{xy^3}-\frac{3^2}{2^{13}}\frac{1}{xy^5}+\frac{3}{2^9}\frac{1}{xy^7}-\frac{1}{2^5}\frac{1}{xy^9}-\frac{3^2}{2^{15}}\frac{1}{x^2y}+\frac{3}{2^{11}}\frac{1}{x^2y^3}-\frac{1}{2^7}\frac{1}{x^2y^5} \\
& +\frac{1}{2^3 3}\frac{1}{x^2y^7}-\frac{1}{2^9}\frac{1}{x^3y}+\frac{1}{2^5 3}\frac{1}{x^3y^3}-\frac{1}{2^3 3^2}\frac{1}{x^4y} \ ].
\end{aligned}
$$

4.4 Z_{11} 型特異点

$f = x^3y + xy^4 + y^5$,

$f_x = 3x^2y + y^4$, $f_y = x^3 + 4xy^3 + 5y^4$.

$\mathrm{Gb} = \{3yx^2 + y^4, x^3 - 15yx^2 + 4y^3x, -11yx^3 - 15y^2x^2, 58564x^5 - 37125x^4 - 759375y^2x^2\}$.

$I = I_1 \cap I_2$,

$I_1 = \langle 121y + 675, 1331x - 10125\rangle$, $\sqrt{I_1} = \langle 121y + 675, 1331x - 10125\rangle$,

$I_2 = \langle 3yx^2 + y^4, x^3 - 15yx^2 + 4y^3x, 11yx^3 + 15y^2x^2, 11x^4 + 225y^2x^2\rangle$, $\sqrt{I_2} = \langle y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, P_1, P_2\rangle$,

$P_1 = (33x^2 + 45yx)\partial_x + (22yx + 30y^2)\partial_y + 187x + 240y$,

$P_2 = (270x^2 + (-792y^2 - 3150y)x - 220y^3)\partial_x + (345yx - 528y^3 - 2475y^2)\partial_y + 1770x - 4488y^2 - 18675y$.

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$,

$$
\begin{aligned}
P = \ & (1621488000x^2+2211120000yx)\partial_x^2 \\
& +(-1507537185x^2-463117050yx+2171748375y^2)\partial_y\partial_x \\
& +(833546736x^2+(-3104767072y-8186767725)x-746016700y^2+17492101875y)\partial_x \\
& +(63147480x^2-1332723150yx-1934772750y^2)\partial_y^2 \\
& +((555697824y-10098876810)x-2041779168y^2-29844557550y)\partial_y \\
& +4723431504x-17355122928y-112679770725.
\end{aligned}
$$

$$\sigma = [\ \frac{11^{11}}{3^{17}5^{12}}\frac{1}{xy} - \frac{11^9}{3^{14}5^{10}}\frac{1}{xy^2} + \frac{11^7}{3^{11}5^8}\frac{1}{xy^3} - \frac{11^5}{3^8 5^6}\frac{1}{xy^4} + \frac{11^3}{3^5 5^4}\frac{1}{xy^5} - \frac{11}{3^2 5^2}\frac{1}{xy^6} + \frac{11^8}{3^{13}5^9}\frac{1}{x^2y} - \frac{11^6}{3^{10}5^7}\frac{1}{x^2y^2}$$
$$+\frac{11^4}{3^7 5^5}\frac{1}{x^2y^3} - \frac{11^2}{3^4 5^3}\frac{1}{x^2y^4} + \frac{1}{3\cdot 5}\frac{1}{x^2y^5} + \frac{11^5}{3^9 5^6}\frac{1}{x^3y} - \frac{11^3}{3^6 5^4}\frac{1}{x^3y^2} + \frac{11}{3^3 5^2}\frac{1}{x^3y^3}$$
$$+\frac{11^2}{3^6 5^3}\frac{1}{x^4y} - \frac{1}{3^2 5}\frac{1}{x^4y^2} - \frac{1}{3}\frac{1}{x^5y}\ ].$$

4.5 Z_{12} 型特異点

$f = x^3y + xy^4 + x^2y^3$,
$f_1 = 3x^2y + y^4 + 2xy^3$, $f_2 = x^3 + 4xy^3 + 3x^2y^2$.
$\mathrm{Gb} = \{3yx^2 + y^4 + 2y^3x, x^3 + (3y^2 - 6y)x^2 - 2y^4, (-14y + 5)x^3 - 30yx^2 - 5y^5 - 10y^4,$
$(105y - 55)x^4 + 484yx^3, 735x^5 + 605x^4 - 5324yx^3\}$.
$I = I_1 \cap I_2$,
$I_1 = \langle 35y - 121, 245x + 1331\rangle$, $\sqrt{I_1} = \langle 35y - 121, 245x + 1331\rangle$,
$I_2 = \langle 3yx^2 + y^4 + 2y^3x, x^3 + (3y^2 - 6y)x^2 - 2y^4, 5x^4 - 44yx^3, (14y - 5)x^3 + 30yx^2 + 5y^5 + 10y^4\rangle$,
$\sqrt{I_2} = \langle y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, P_1, P_2\rangle$,

$$\begin{aligned} P_1 = \ & (140x^2 + 248yx + 44y^2)\partial_x + ((70y + 33)x + 125y^2)\partial_y + 770x + 1158y, \\ P_2 = \ & ((84y + 124)x^2 + (-672y^2 + 2770y)x - 42y^3 - 176y^2)\partial_x \\ & +((42y^2 - 169y - 132)x - 609y^3 + 2008y^2)\partial_y + (462y + 340)x - 4914y^2 + 16686y. \end{aligned}$$

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$,

$$\begin{aligned} P = \ & (194685857871120x^2 + 420775133985360yx + 180463194172800y^2)\partial_x^2 \\ & +(-82832632888320x^2 + (-9751678231680y - 180463194172800)x + 107193098960640y^2)\partial_y\partial_x \\ & +(-133092828502732x^3 + (1373829024443004y - 3548483873971782)x^2 \\ & \quad +(-315326509783170y^2 + 2838895228228440y + 76535249452200)x \\ & \quad -696421142828840y^3 - 81448893497400y^2 + 1647347302339200y)\partial_x \\ & +((-92078741076840y + 29963270244600)x - 70175382520320y^2 - 210540393201600y)\partial_y^2 \\ & +((-66546414251366y + 350146787028213)x^2 \\ & \quad +(1071890730792481y^2 - 3093414693538428y - 755979214347180)x \\ & \quad -325846150902008y^3 + 2066998836578100y^2 - 1513707145196400y - 1864786339785600)\partial_y \\ & -732010556765026x^2 + (776512086574553 8y - 20427450896819658)x \\ & -3303190350044904y^2 + 17240073586133520y - 6930858351546360. \end{aligned}$$

$$\sigma = [\ \frac{5^5 7^7}{11^{13}}\frac{1}{xy} + \frac{5^4 7^6}{11^{11}}\frac{1}{xy^2} + \frac{5^3 7^5}{11^9}\frac{1}{xy^3} + \frac{5^2 7^4}{11^7}\frac{1}{xy^4} + \frac{5\cdot 7^3}{11^5}\frac{1}{xy^5} + \frac{7^2}{11^3}\frac{1}{xy^6} - \frac{3}{11}\frac{1}{xy^7}$$
$$-\frac{5^4 7^5}{11^{10}}\frac{1}{x^2y} - \frac{5^3 7^4}{11^8}\frac{1}{x^2y^2} - \frac{5^2 7^3}{11^6}\frac{1}{x^2y^3} - \frac{5\cdot 7^2}{11^4}\frac{1}{x^2y^4} - \frac{7}{11^2}\frac{1}{x^2y^5}$$
$$+\frac{5^3 7^3}{11^7}\frac{1}{x^3y} + \frac{5^2 7^2}{11^5}\frac{1}{x^3y^2} + \frac{5\cdot 7}{11^3}\frac{1}{x^3y^3} + \frac{1}{11}\frac{1}{x^3y^4} - \frac{5^2 7}{11^4}\frac{1}{x^4y} - \frac{5}{11^2}\frac{1}{x^4y^2} - \frac{2^2}{11}\frac{1}{x^5y}\ ].$$

4.6 Z_{13} 型特異点

$f = x^3y + y^6 + xy^5$,
$f_x = 3x^2y + y^5$, $f_2 = x^3 + 6y^5 + 5xy^4$.
$\mathrm{Gb} = \{-7yx^3 - 9y^2x^2, 3yx^2 + y^5, x^3 - 18yx^2 + 5y^4x, -343x^5 - 729y^3x^2\}$.
$I = I_1 \cap I_2$,
$I_1 = \langle 7x + 9y, 49y^2 + 243\rangle$, $\sqrt{I_1} = \langle 49y^2 + 243, 7x + 9y\rangle$,
$I_2 = \langle 7yx^3 + 9y^2x^2, 7x^4 + 162y^2x^2, 3yx^2 + y^5, x^3 - 18yx^2 + 5y^4x, y^3x^2\rangle$, $\sqrt{I_2} = \langle y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, P_1, P_2\rangle$,

$P_1 = (14x^2 + 18yx)\partial_x + (7yx + 9y^2)\partial_y + 77x + 90y$,
$P_2 = (72x^2 + (-280y^3 - 891y)x - 105y^4)\partial_x + (99yx - 140y^4 - 567y^2)\partial_y + 486x - 1540y^3 - 5184y$.

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$,

$$
\begin{aligned}
P = \ & (3662593200x^2 + 4709048400yx)\partial_x^2 \\
& +(-2683171008x^2 + 1890641088yx + 6866270208y^2)\partial_y\partial_x \\
& +(-42710470250x^3 - 105737060100yx^2 + (-97769239650y^2 - 90813857211)x \\
& -7368256665y^3 + 55324766448y)\partial_x \\
& +(262919790x^2 - 3426024924yx - 4839511698y^2)\partial_y^2 \\
& +(-21355235125yx^2 + (-52868530050y^2 - 19843900587)x - 48884619825y^3 - 1076714 \\
& -234907586375x^2 - 554097099675yx - 502429306950y^2 - 667382260608.
\end{aligned}
$$

$$
\begin{aligned}
\sigma = [\ & \frac{7^7}{2\cdot 3^{19}}\frac{1}{xy} - \frac{7^5}{2\cdot 3^{14}}\frac{1}{xy^3} + \frac{7^3}{2\cdot 3^9}\frac{1}{xy^5} - \frac{7}{2\cdot 3^4}\frac{1}{xy^7} + \frac{7^4}{2\cdot 3^{12}}\frac{1}{x^2y^2} - \frac{7^2}{2\cdot 3^7}\frac{1}{x^2y^4} + \frac{1}{2\cdot 3^2}\frac{1}{x^2y^6} \\
& -\frac{7^3}{2\cdot 3^{10}}\frac{1}{x^3y} + \frac{7}{2\cdot 3^5}\frac{1}{x^3y^3} - \frac{1}{2\cdot 3^3}\frac{1}{x^4y^2} - \frac{1}{3}\frac{1}{x^5y}\ ].
\end{aligned}
$$

4.7 W_{12} 型特異点

$f = x^4 + y^5 + x^2y^3$,
$f_x = 4x^3 + 2xy^4$, $f_y = 5y^4 + 3x^2y^2$.
$\mathrm{Gb} = \{3y^2x^2 + 5y^4, 2x^3 + y^4x, -9x^5 - 50x^3, 6x^4 - 5y^6\}$.
$I = I_1 \cap I_2$,
$I_1 = \langle 3y - 10, 27x^2 + 500\rangle$, $\sqrt{I_1} = \langle 3y - 10, 27x^2 + 500\rangle$,
$I_2 = \langle 2x^3 + y^3x, 3y^2x^2 + 5y^4, yx^3, 6x^4 - 5y^5, x^5\rangle$, $\sqrt{I_2} = \langle y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, P_1, P_2\rangle$,
$P_1 = ((12y - 30)x^2 + 5y^3)\partial_x + (9y^2 - 30y)x\partial_y + (78y - 210)x$,
$P_2 = (12x^3 + (5y^2 + 50y)x)\partial_x + (9yx^2 + 50y^2)\partial_y + 78x^2 + 350y$.

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$,

$$
\begin{aligned}
P = \ & (12000x^2 + 20000y^2)\partial_x^2 + (144000y - 480000)x\partial_y\partial_x \\
& +(-291519x^3 + (-81675y^2 - 1009800y - 933000)x)\partial_x \\
& +(-45000x^2 + 69000y^2 - 480000y)\partial_y^2 \\
& +((-210681y + 106110)x^2 - 723600y^2 + 240000y - 3840000)\partial_y \\
& -1878957x^2 - 5788800y - 5499000.
\end{aligned}
$$

$$
\begin{aligned}
\sigma = [\ & -\frac{3^6}{10^7}\frac{1}{xy} - \frac{2^5}{10^6}\frac{1}{xy^2} - \frac{3^4}{10^5}\frac{1}{xy^3} - \frac{3^3}{10^4}\frac{1}{xy^4} - \frac{3^2}{10^3}\frac{1}{xy^5} - \frac{3}{10^2}\frac{1}{xy^6} \\
& +\frac{3^3}{2\cdot 10^4}\frac{1}{x^3y} + \frac{3^2}{2\cdot 10^3}\frac{1}{x^3y^2} + \frac{3}{2\cdot 10^2}\frac{1}{x^3y^3} + \frac{1}{2\cdot 10}\frac{1}{x^3y^4} - \frac{1}{2^2 10}\frac{1}{x^5y}\ ].
\end{aligned}
$$

4.8 W_{13} 型特異点

$f = x^4 + xy^4 + y^6$,
$f_x = 4x^3 + y^4$, $f_y = 4xy^3 + 6y^5$.
$\mathrm{Gb} = \{4x^3 + y^4, -6yx^3 + y^3x, -9x^6 - x^5\}$.
$I = I_1 \cap I_2$,
$I_1 = \langle 9x + 1, 27y^2 - 2\rangle$, $\sqrt{I_1} = \langle 27y^2 - 2, 9x + 1\rangle$,
$I_2 = \langle 4x^3 + y^4, 6yx^3 - y^3x, y^3x^2, x^5\rangle$, $\sqrt{I_2} = \langle y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, P_1, P_2\rangle$,
$P_1 = (2x^2 + 3y^2x)\partial_x + 8x + 9y^2$,
$P_2 = (270yx^2 + 8yx - 33y^3)\partial_x + (-36x^2 + 6y^2)\partial_y + 810yx + 50y$.

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, P\rangle$,

$$
\begin{aligned}
P = \ & (4032x^2 + (-5778y^2 + 896)x + 2673y^4 - 168y^2)\partial_x^2 + (4512yx - 2304y^3 + 672y)\partial_x\partial_y \\
& +((192456y^2 + 34272)x - 31068y^2 + 6496)\partial_x + (336x + 504y^2)\partial_y^2 + (96228y^3 - 600y)\partial_y \\
& +1058508y^2 - 22680.
\end{aligned}
$$

$$
\begin{aligned}
\sigma = [\ & -\frac{3^9}{2^5}\frac{1}{xy} - \frac{3^6}{2^4}\frac{1}{xy^3} - \frac{3^3}{2^3}\frac{1}{xy^5} - \frac{1}{2^2}\frac{1}{xy^7} + \frac{3^7}{2^5}\frac{1}{x^2y} + \frac{3^4}{2^4}\frac{1}{x^2y^3} + \frac{3}{2^3}\frac{1}{x^2y^5} - \frac{3^5}{2^5}\frac{1}{x^3y} \\
& -\frac{3^2}{2^4}\frac{1}{x^3y^3} + \frac{3^3}{2^5}\frac{1}{x^4y} + \frac{1}{2^4}\frac{1}{x^4y^3} - \frac{3}{2^5}\frac{1}{x^5y}\ ].
\end{aligned}
$$

4.9 Q_{10} 型特異点

$f = x^3 + y^4 + yz^2 + xy^3$,
$f_x = 3x^2 + y^3$, $f_y = 4y^3 + z^2 + 3xy^2$, $f_z = 2yz$.
$\mathrm{Gb} = \{zy, z^3, 3x^2 + y^3, -12x^2 + 3y^2x + z^2, zx^2, -3x^3 - 4yx^2\}$.
$I = I_1 \cap I_2$,
$I_1 = \langle zy, z^3, zx^2, 3x^2 + y^3, 12x^2 - 3y^2x - z^2, 16yx^2 + z^2x, 12x^3 - z^2x\rangle$, $\sqrt{I_1} = \langle z, y, x\rangle$,
$I_2 = \langle z, 3y + 16, 9x - 64\rangle$, $\sqrt{I_2} = \langle z, 3y + 16, 9x - 64\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, f_z, P_1, P_2, P_3\rangle$,

$$\begin{aligned}
P_1 = {} & 2zx\partial_x + 2zy\partial_y + 3z^2\partial_z + 15z,\\
P_2 = {} & (18x^2 + 24yx)\partial_x + (12yx + 16y^2)\partial_y + (21zx + 24zy)\partial_z + 111x + 136y,\\
P_3 = {} & (162x^3 + (72y - 1152)x^2 - 512yx - 32z^2)\partial_x + (108yx^2 - 768yx)\partial_y + (189zx^2 - 1344zx)\partial_z\\
& +999x^2 + (144y - 7104)x + 192y^2 - 1024y.
\end{aligned}$$

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, f_z, P\rangle$,

$$\begin{aligned}
P = {} & (1536x - 384y^2)\partial_x^2 + (768x + 1024y)\partial_y\partial_x + 1536z\partial_z\partial_x + (-1458x - 1080y + 10240)\partial_x\\
& +(432x + 486y^2 + 3168y)\partial_y^2 + 4224z\partial_z\partial_y + (3888y + 21696)\partial_y\\
& +(1296x^2 - 2304y^2)\partial_z^2 + 729z\partial_z + 2187.
\end{aligned}$$

$$\begin{aligned}
\sigma = [\quad & -\frac{3^4}{2^{21}}\frac{1}{xyz} + \frac{3^3}{2^{17}}\frac{1}{xy^2z} - \frac{3^2}{2^{18}}\frac{1}{xy^3z} + \frac{3}{2^9}\frac{1}{xy^4z} - \frac{1}{2^5}\frac{1}{xy^5z} - \frac{3^2}{2^{15}}\frac{1}{x^2yz} - \frac{1}{2\cdot 3}\frac{1}{x^2yz^3} + \frac{3}{2^{11}}\frac{1}{x^2y^2z}\\
& -\frac{1}{2^7}\frac{1}{x^2y^3z} + \frac{1}{2^3 3}\frac{1}{x^2y^4z} - \frac{1}{2^9}\frac{1}{x^3yz} + \frac{1}{2^5 3}\frac{1}{x^3y^2z} - \frac{1}{2^3 3^2}\frac{1}{x^4yz}].
\end{aligned}$$

4.10 Q_{11} 型特異点

$f = x^3 + y^2z + xz^3 + z^5$
$f_x = 3x^2 + z^3$, $f_y = 2yz$, $f_z = 3xz^2 + y^2 + 5z^4$.
$\mathrm{Gb} = \{zy, 3x^2 + z^3, y^3, -15zx^2 + 3z^2x + y^2, yx^2, 375x^4 + 9x^3 - 5y^2x\}$.
$I = I_1 \cap I_2$,
$I_1 = \langle 25z + 3, y, 125x + 3\rangle$, $\sqrt{I_1} = \langle 25z + 3, y, 125x + 3\rangle$,
$I_2 = \langle zy, 3x^2 + z^3, 15zx^2 - 3z^2x - y^2, y^3, yx^2, 9x^3 - 5y^2x\rangle$, $\sqrt{I_2} = \langle z, y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, f_z, P_1, P_2, P_3\rangle$,

$$\begin{aligned}
P_1 &= (120x^2 - 18zx + 10z^3)\partial_x + (120yx - 21zy)\partial_y + (60zx - 12z^2)\partial_z + 660x - 111z,\\
P_2 &= ((400z^2 + 18z)x - 50z^3)\partial_x + 21zy\partial_y + (-60zx + 12z^2)\partial_z - 60x + 800z^2 + 111z,\\
P_3 &= ((400z + 18)yx - 50z^2y)\partial_x + (-180zx^2 - 300z^3x + 21y^2)\partial_y + (-60yx + 12yz)\partial_z + 111y.
\end{aligned}$$

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, f_z, P\rangle$,

$$\begin{aligned}
P = {} & (162x + 270z^2)\partial_z\partial_x + (-1500x + 840z)\partial_x + (1425z^2 + 171z)x\partial_y^2 + 246y\partial_z\partial_y + 750y\partial_y\\
& +(285x + 1375z^2 + 108z)\partial_z^2 + (10000z + 1164)\partial_z + 6500.
\end{aligned}$$

$$\begin{aligned}
\sigma = [\quad & \frac{5^{10}}{2\cdot 3^6}\frac{1}{xyz} - \frac{5^8}{2\cdot 3^5}\frac{1}{xyz^2} + \frac{5^6}{2\cdot 3^4}\frac{1}{xyz^3} - \frac{5^4}{2\cdot 3^3}\frac{1}{xyz^4} + \frac{5^2}{2\cdot 3^2}\frac{1}{xyz^5} - \frac{1}{2\cdot 3}\frac{1}{xyz^6}\\
& -\frac{5^7}{2\cdot 3^5}\frac{1}{x^2yz} + \frac{5^5}{2\cdot 3^4}\frac{1}{x^2yz^2} - \frac{5^3}{2\cdot 3^3}\frac{1}{x^2yz^3} + \frac{5}{2\cdot 3^2}\frac{1}{x^2yz^4} - \frac{1}{2\cdot 3}\frac{1}{x^2y^3z}\\
& +\frac{5^4}{162}\frac{1}{x^3yz} - \frac{5^2}{2\cdot 3^3}\frac{1}{x^3yz^2} + \frac{1}{2\cdot 3^2}\frac{1}{x^3yz^3} - \frac{5}{2\cdot 3^3}\frac{1}{x^4yz}].
\end{aligned}$$

4.11 Q_{12} 型特異点

$f = x^3 + y^5 + yz^2 + xy^4$,
$f_x = 3x^2 + y^4$, $f_y = 5y^4 + z^2 + 4xy^3$, $f_z = 2yz$.
$\mathrm{Gb} = \{zy, z^3, zx^2, -4x^3 - 5yx^2, 3x^2 + y^4, -15x^2 + 4y^3x + z^2\}$
$I = I_1 \cap I_2$,
$I_1 = \langle zy, z^3, zx^2, 75yx^2 + 4z^2x, 15x^3 - z^2x, 3x^2 + y^4, 15x^2 - 4y^3x - z^2\rangle$, $\sqrt{I_1} = \langle z, y, x\rangle$,
$I_2 = \langle z, 4x + 5y, 16y^2 + 75\rangle$, $\sqrt{I_2} = \langle z, 4x + 5y, 16y^2 + 75\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, f_z, P_1, P_2, P_3\rangle$,

$$\begin{aligned} P_1 = {} & zx\partial_x + zy\partial_y + 2z^2\partial_z + 9z, \\ P_2 = {} & (8x^2+10yx)\partial_x + (4yx+5y^2)\partial_y + (10zx+10zy)\partial_z + 50x + 55y, \\ P_3 = {} & ((192y^2+600)x^2 - 375yx - 20z^2)\partial_x + (-120x^2+96y^3x-375y^2)\partial_y \\ & +(240zy^2x - 40zy^3 - 750zy)\partial_z + (1200y^2+1200)x - 3375y. \end{aligned}$$

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, f_z, P\rangle$,

$$\begin{aligned} P = {} & (3750x-1000y^3)\partial_x^2 + (1500x+1875y)\partial_y\partial_x + 3750z\partial_z\partial_x + (-4608yx - 3360y^2 + 24375)\partial_x \\ & +(720x+1152y^3+6300y)\partial_y^2 + 11850z\partial_z\partial_y + (12672y^2+52950)\partial_y \\ & +(2880yx^2 - 4500y^3)\partial_z^2 + 13824y. \end{aligned}$$

$$\begin{aligned} \sigma = [\ & -\frac{2^9}{3^3 5^6}\frac{1}{xy^2z} + \frac{2^5}{3^2 5^4}\frac{1}{xy^4z} - \frac{2}{3\cdot 5^2}\frac{1}{xy^6z} + \frac{2^7}{3^3 5^5}\frac{1}{x^2yz} - \frac{1}{2\cdot 3}\frac{1}{x^2yz^3} - \frac{2^3}{3^2 5^3}\frac{1}{x^2y^3z} \\ & + \frac{1}{2\cdot 3\cdot 5}\frac{1}{x^2y^5z} + \frac{2}{3^2 5^2}\frac{1}{x^3y^2z} - \frac{1}{2\cdot 3^2 5}\frac{1}{x^4yz}]. \end{aligned}$$

4.12 S_{11} 型特異点

$f = x^4 + y^2z + xz^2 + x^3z$,

$f_x = 4x^3 + z^2 + 3x^2z,\ f_y = 2yz,\ f_z = y^2 + 2xz + x^3$.

$$\begin{aligned} \mathrm{Gb} = {} & \{zy, y^3, (12y^2+15z^2+64z)x + 32y^2 - 8z^2, 3zx^2 - 8zx - 4y^2 + z^2, x^3 + 2zx + y^2, \\ & -512z^2x + 75z^4 + 64z^3, (-15z^3 - 64z^2)x + 8z^3\}. \end{aligned}$$

$I = I_1 \cap I_2$,

$I_1 = \langle 25z+128, y, 5x-16\rangle,\ \sqrt{I_1} = \langle 25z+128, y, 5x-16\rangle$,

$I_2 = \langle zy, 8z^2x - z^3, 3zx^2 - 8zx - 4y^2 + z^2, y^3, (96y^2+512z)x + 256y^2 + 15z^3 - 64z^2, x^3 + 2zx + y^2, z^4\rangle$,

$\sqrt{I_2} = \langle z, y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, f_z, P_1, P_2, P_3\rangle$,

$$\begin{aligned} P_1 &= (8yx+2zy)\partial_x + 4y^2\partial_y + 3zyx\partial_z + 3yx + 36y, \\ P_2 &= (64x^2+56zx+10z^2)\partial_x + (-64xy+20zy)\partial_y + (24zx^2+15z^2x+96y^2)\partial_z + 24x^2 + 15zx + 84z, \\ P_3 &= (16xz+10z^2)\partial_x + (-96xy-16yz)\partial_y + (15xz^2+96y^2-48z^2)\partial_z + (15z-288)x - 168z. \end{aligned}$$

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, f_z, P\rangle$,

$$\begin{aligned} P = {} & (480x+300z)\partial_x^2 - 288y\partial_x\partial_y + (-2048x - 1280z)\partial_z + (-144x - 90z - 608)\partial_x \\ & +(-360zx + 1152z)\partial_y^2 - 2560y\partial_y\partial_z + 144y\partial_y + (-2304x^2 - 1344zx - 540z^2 - 3072z)\partial_z^2 \\ & +((-135z - 2112)x - 3168z - 19968)\partial_z - 135x - 2088. \end{aligned}$$

$$\begin{aligned} \sigma = [\ & -\frac{5^5}{2^{23}}\frac{1}{xyz} + \frac{5^3}{2^{16}}\frac{1}{xyz^2} - \frac{5}{2^9}\frac{1}{xyz^3} - \frac{1}{2^2}\frac{1}{xyz^4} - \frac{5^4}{2^{19}}\frac{1}{x^2yz} + \frac{5^2}{2^{12}}\frac{1}{x^2yz^2} - \frac{1}{2^5}\frac{1}{x^2yz^2} - \frac{5^3}{2^{15}}\frac{1}{x^3y^3z} \\ & + \frac{5}{2^8}\frac{1}{x^3yz^2} - \frac{1}{2^3}\frac{1}{x^3y^3z} - \frac{5^2}{2^{11}}\frac{1}{x^4yz} + \frac{1}{2^4}\frac{1}{x^4yz^2} - \frac{5}{2^7}\frac{1}{x^5yz}]. \end{aligned}$$

4.13 S_{12} 型特異点

$f = x^2y + y^2z + xz^3 + z^5$,

$f_x = 2xy + z^3,\ f_y = x^2 + 2yz,\ f_z = y^2 + 3xz^2 + 5z^4$.

$$\begin{aligned} \mathrm{Gb} = {} & \{x^2 + 2zy, 2yx + z^3, (-10zy + 3z^2)x + y^2, -13y^2x - 20z^2y^2, -169y^2x - 800zy^3, 2197y^2x + 32000y^4, \\ & -4000y^3x + 130y^3 + 507zy^2\}. \end{aligned}$$

$I = I_1 \cap I_2$,

$I_1 = \langle 40y - 13z, 400x - 169, 8000z^2 + 2197\rangle,\ \sqrt{I_1} = \langle 8000z^2 + 2197, 40y - 13z, 400x - 169\rangle$,

$I_2 = \langle x^2 + 2zyz, 2yx + z^3, (10zy - 3z^2)x - y^2, 10y^3 + 39zy^2, y^2x, z^2y^2\rangle,\ \sqrt{I_2} = \langle z, y, x\rangle$.

$\mathcal{A}nn^{(1)} = \langle f_x, f_y, f_z, P_1, P_2\rangle$,

$$\begin{aligned} P_1 = {} & (104zx + 1800z^2y - 425z^3)\partial_x + ((600zy - 295z^2)x + 130zy)\partial_y + (-240zy + 78z^2)\partial_z \\ & -300zx - 30y + 702z, \\ P_2 = {} & ((17000zy - 3000z^2)x - 1100y^2 - 7605zy + 1521z^2)\partial_x \\ & +((3000y - 3000z)x^2 + (-2535y + 1521z)x - 750z^2y)\partial_y \\ & +(-2400yx + 1014y)\partial_z + 1170x + 11000zy + 1950z^2. \end{aligned}$$

$\mathcal{A}nn^{(2)} = \langle f_x, f_y, f_z, P\rangle$,

$$P = (4732x+7280z^2)\partial_x^2+(-4550zx+5915y)\partial_x\partial_y+(-28700zx+26390y+3549z)\partial_x\partial_z$$
$$+(904300x+1500000z^2+36673)\partial_x+(-12600zy+4095z^2)\partial_y\partial_z$$
$$+((-9000000y+2055000z)x+1244400y-37310z)\partial_y+(9555x-90000zy+43950z^2)\partial_z^2$$
$$+(-3000000zx+1020000y+1012000z)\partial_z-21000000x+6916600.$$

$$\sigma=[\ -\frac{2^{13}5^6}{13^7}\frac{1}{xyz^2}+\frac{2^75^3}{13^4}\frac{1}{xyz^4}-\frac{2}{13}\frac{1}{xyz^6}-\frac{2^{10}5^5}{13^6}\frac{1}{xy^2z}+\frac{2^45^2}{13^3}\frac{1}{xy^2z^3}+\frac{2\cdot5}{13^2}\frac{1}{xy^3z^2}-\frac{3}{13}\frac{1}{xy^4z}$$
$$-\frac{2^95^4}{13^5}\frac{1}{x^2yz^2}+\frac{2^35}{13^2}\frac{1}{x^2yz^2}-\frac{2^65^3}{13^4}\frac{1}{x^2y^2z}+\frac{1}{13}\frac{1}{x^2y^2z^3}-\frac{2^55^2}{13^3}\frac{1}{x^3yz^2}-\frac{2^25}{13^2}\frac{1}{x^3y^2z}-\frac{2}{13}\frac{1}{x^4yz^2}\ ].$$

4.14 U_{12} 型特異点

$f=x^3+y^3+z^4+xyz^2,$

$f_x=3x^2+yz^2,\ f_y=3y^2+xz^2,\ f_z=4z^3+2xyz.$

$\mathrm{Gb}=\{3x^2+z^2y, z^2x+3y^2, zyx+2z^3, -x^3+y^3, -3y^3+2z^4, -6y^2x+y^4, -6yx^2+y^3x\}.$

$I=I_1\cap I_2\cap I_3,$

$I_1=\langle x+y+6, z^2+18, y^2+6y+36\rangle,\ \sqrt{I_1}=\langle z^2+18, x+y+6, y^2+6y+36\rangle,$

$I_2=\langle y-6, x-6, z^2+18\rangle,\ \sqrt{I_2}=\langle z^2+18, y-6, x-6\rangle,$

$I_3=\langle 3x^2+z^2y, z^2x+3y^2, zy^2, zyx+2z^3, zx^2, y^2x, yx^2, x^3-y^3, 3y^3-2z^4, y^4\rangle,\ \sqrt{I_3}=\langle z,y,x\rangle.$

$\mathcal{A}nn^{(1)}=\langle f_x,f_y,f_z,P_1,P_2,P_3\rangle,$

$$P_1 = (48x^2-11y^2x-6z^2y)\partial_x+(60yx-10y^3)\partial_y+(36zx-6zy^2)\partial_z+324x-69y^2,$$
$$P_2 = (4x^3-24yx)\partial_x+(5yx^2+6z^2x-12y^2)\partial_y+(3zx^2-18zy)\partial_z+30x^2-126y,$$
$$P_3 = (26zyx^2-576zx-60zy^2)\partial_x+(29zy^2x+(18z^3-720z)y)\partial_y+((16z^2+72)yx-432z^2)\partial_z$$
$$-362z^3-3888z.$$

$\mathcal{A}nn^{(2)}=\langle f_x,f_y,f_z,P\rangle,$

$$P = (864x-144y^2)\partial_x\partial_y+(-2x^2-276y)\partial_x+(-144x^2+864y)\partial_y^2+(24z^3+432z)\partial_y\partial_z$$
$$+(yx+218z^2+5616)\partial_y+(-36x+6y^2)\partial_z^2-3x.$$

$$\sigma=[\ -\frac{1}{2^53^6}\frac{1}{xyz}+\frac{1}{2^43^4}\frac{1}{xyz^3}-\frac{1}{2^33^2}\frac{1}{xyz^5}-\frac{1}{2^23^3}\frac{1}{xy^4z}-\frac{1}{2^33^4}\frac{1}{x^2y^2z}+\frac{1}{2^23^2}\frac{1}{x^2y^2z^3}-\frac{1}{2^23^3}\frac{1}{x^4yz}\ ].$$

5 Bimodal singularity E_{18} に関する計算

2 変数半擬斉次多項式 $f(x,y)=x^3+y^{10}+xy^7+xy^8$ の定める E_{18} 型 Bimodal 特異点に対し, $\mathcal{A}nn^{(1)}$ を求め, §4 と同様の計算を行った.

$\mathcal{A}nn^{(1)}$ を用いて代数的局所コホモロジー類 $[1/f_xf_y]|_{(0,0)}$ を計算すると, 次の表現を得る.

$$[\,a\frac{1}{xy}+b\frac{1}{xy^2}+\frac{523262749825378986255543}{3^{10}10^{20}}\frac{1}{xy^3}+\frac{3542859151064368070093}{3^910^{18}}\frac{1}{xy^4}$$
$$+\frac{272111019228806143}{3^810^{16}}\frac{1}{xy^5}-\frac{3787973405502307}{3^710^{14}}\frac{1}{xy^6}-\frac{13210502493257}{3^610^{12}}\frac{1}{xy^7}$$
$$+\frac{14232708293}{3^510^{10}}\frac{1}{xy^8}+\frac{188727343}{3^410^8}\frac{1}{xy^9}+\frac{438893}{3^310^6}\frac{1}{xy^{10}}-\frac{2057}{3^210^4}\frac{1}{xy^{11}}$$
$$-\frac{7}{3\cdot10^2}\frac{1}{xy^{12}}-\frac{24652664985303778499}{3^910^{17}}\frac{1}{x^2y}-\frac{757155250872126449}{3^810^{15}}\frac{1}{x^2y^2}$$
$$+\frac{107457356825701}{3^710^{13}}\frac{1}{x^2y^3}+\frac{1264904961551}{3^610^{11}}\frac{1}{x^2y^4}+\frac{1815069901}{3^510^9}\frac{1}{x^2y^5}$$
$$-\frac{11224249}{3^410^7}\frac{1}{x^2y^6}-\frac{45899}{3^310^5}\frac{1}{x^2y^7}-\frac{49}{3^210^3}\frac{1}{x^2y^8}+\frac{1}{30}\frac{1}{x^2y^9}-\frac{70850911193}{3^610^{10}}\frac{1}{x^3y}$$
$$-\frac{320395243}{3^510^8}\frac{1}{x^3y^2}+\frac{178207}{3^410^6}\frac{1}{x^3y^3}+\frac{4157}{3^310^4}\frac{1}{x^3y^4}+\frac{7}{3^210^2}\frac{1}{x^3y^5}-\frac{251}{3^310^3}\frac{1}{x^4y}-\frac{1}{3^210}\frac{1}{x^4y^2}\,].$$

E_{18} の場合, 1 階の微分作用素を用いた計算では $[1/xy]$ と $[1/xy^2]$ の係数 a, b を決めることはできないことが分かる. まとめると, 次の (i), (ii) を得る.

(i) ホロノミック系 $\mathcal{D}_X/\mathcal{A}nn^{(1)}$ の原点における重複度 $=3$

(ii) $Hom_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{A}nn^{(1)}, \mathcal{H}^2_{[(0,0)]}(\mathcal{O}_X))=\mathrm{Span}\{[1/xy],[1/xy^2],\sigma\}$

このことは, 特異点の"複雑さ"とホロノミック系 $\mathcal{D}_X/\mathcal{A}nn^{(1)}$ の重複度の間には何らかの関係があることを示唆していると思われる.

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